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QED is not endangered by the proton's size

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ABSTRACT

Pohl et al. have reported a very precise measurement of the Lamb-shift in muonic hydrogen (Pohl et al., 2010) [1], from which they infer the radius characterizing the proton's charge distribution. The result is 5 standard deviations away from the one of the CODATA compilation of physical constants. This has been interpreted (Pohl et al., 2010) [1] as possibly requiring a 4.9 standard-deviation modification of the Rydberg constant, to a new value that would be precise to 3.3 parts in 10^{13}, as well as putative evidence for physics beyond the standard model (Flowers, 2010) [2]. I demonstrate that these options are unsubstantiated.

1. Introduction

The issue is extremely simple. The discrepancy quoted in the abstract is between results which do not depend on a specific model of the proton's form factor and results, by Pohl et al., which do [3]. The conclusion is not that the experiments or the theory are wrong, but that the model (the customary dipole form factor) is inadequate at the level of precision demanded by the data. The experiments and QED are right, the dipole is wrong. More generally, it is risky to use a model of the charge distribution to extract a property of the very same charge distribution.

The conclusion of the previous paragraph is the expected one. The dipole form factor is but a rough description of higher-energy data and is unacceptable on grounds of the analyticity requirements stemming from causality and the locality of fundamental interactions.

Moreover, any simple one-parameter description of the proton's non-relativistic Sacks form factor, $G_E(-\mathbf{q}^2)$ in terms of only one mass parameter is inaccurate: the proton is not so simple. More precisely, the proton's relativistic form factor, $G_E(q^2)$, is expected, in the timelike domain $q^2 \geqslant 0$, to have a complex structure, with a first cut starting at $q^2 = 4m_\pi^2$ and a plethora of branch cuts and complex resonant poles thereafter [4].

The same is true of the charge distribution, $\rho_p(r)$, the Fourier transform of $G_E(-\mathbf{q}^2)$. Even most naively, $\rho_p(r)$ is expected to have a "core" and a "pion cloud" [5], corresponding to a minimum of two length parameters.

2. In detail

Let ℓ denote an electron or a μ^-. The leading proton-size correction to the energy levels of an ℓp atom is

$$\Delta E = \frac{2\alpha^4}{3n^3} m_r^3 \delta_{l0} \langle r_p^2 \rangle,$$
$$m_r \equiv \frac{m_\ell m_p}{m_\ell + m_p} \quad (1)$$

where $\langle r_p^2 \rangle$ is the mean square radius of $\rho_p(r)$.

The charge distribution is related to the non-relativistic limit of the electric form-factor, G_E, by the Fourier transformation

$$G_E(-\mathbf{q}^2) = \int d^3r\, \rho_p(\mathbf{r}) e^{-i\vec{q}\vec{r}}. \quad (2)$$

Precise measurements of $\langle r_p^2 \rangle$ have two origins. One is mainly based on the theory [6] and observations [7] of hydrogen. The result, compiled in CODATA [8], is

$$\langle r_p^2 \rangle(\text{CODATA}) = (0.8768 \pm 0.0069\ \text{f})^2. \quad (3)$$

The second type of measurement is based on the theory and observations [9,10] of very low-energy electron–proton scattering. It yields

$$\langle r_p^2 \rangle(ep) = (0.895 \pm 0.018\ \text{f})^2. \quad (4)$$

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This result requires a sophisticated data analysis, partly based on a continued-fraction expansion of G_E [9].

The two quoted methods of measuring $\langle r_p^2\rangle$ are model-independent, in the sense of not assuming a particular form of the proton's charge distribution, $\rho_p(r)$.

The plot thickens as one considers the Lamb shift $2P_{3/2}^{F=2} \to 2S_{1/2}^{F=1}$ in the μp atom, measured [1] to be

$$L_{\exp} = 206.2949 \pm 0.0032 \text{ meV}. \tag{5}$$

In meV units for energy and fermi units for the radii, the predicted value [11] is of the form

$$L^{\rm th}\big[\langle r_p^2\rangle, \langle r_p^3\rangle_{(2)}\big] = 209.9779(49) - 5.2262\langle r_p^2\rangle + 0.00913\langle r_p^3\rangle_{(2)}. \tag{6}$$

The first two coefficients are best estimates of many contributions while the third stems from the $n = 2$ value of an addend [12,6]

$$\Delta E_3(n) = \frac{\alpha^5}{3n^3} m_r^4 \delta_{l0} \langle r_p^3\rangle_{(2)}, \tag{7}$$

proportional to the third Zemach moment

$$\langle r_p^3\rangle_{(2)} \equiv \int d^3r_1\, d^3r_2\, \rho(r_1)\rho(r_2)|\mathbf{r}_1 - \mathbf{r}_2|^3. \tag{8}$$

For a single-parameter description of the charge distribution, there is an explicit relation between $\langle r_p^3\rangle_{(2)}$ and $\langle r_p^2\rangle$. Consider, as an example, a ρ-dominated form factor in its narrow-width non-relativistic limit

$$G_E(q^2) = \frac{m_\rho^2}{q^2 - m_\rho^2 + i m_\rho \Gamma_\rho} \to \frac{m_\rho^2}{\mathbf{q}^2 + m_\rho^2}. \tag{9}$$

The corresponding charge distribution is a Yukawian

$$\rho(r) = \frac{m_\rho^2}{4\pi r} e^{-m_\rho r}. \tag{10}$$

Its relevant moments are $\langle r^0\rangle = 1$,

$$\langle r^2\rangle = 6/m_\rho^2, \qquad \langle r^3\rangle = 24/m_\rho^3, \qquad \langle r^3\rangle_{(2)} = 60/m_\rho^3. \tag{11}$$

The model-dependent relation is thus

$$\big[\langle r^3\rangle_{(2)}\big]^2 = \frac{50}{3}\big[\langle r^2\rangle\big]^3. \tag{12}$$

For a dipole form factor

$$G_E(-\mathbf{q}^2) = \frac{m_d^4}{(\mathbf{q}^2 + m_d^2)^2} \tag{13}$$

the charge distribution is an exponential

$$\rho(r) = \frac{m_d^3}{8\pi} e^{-m_d r} \tag{14}$$

for which $\langle r^0\rangle = 1$,

$$\langle r^2\rangle = 12/m_d^2, \qquad \langle r^3\rangle = 60/m_d^3, \qquad \langle r^3\rangle_{(2)} = 315/(2m_d^3). \tag{15}$$

The model-dependent relation is thus

$$\big[\langle r^3\rangle_{(2)}\big]^2 = \frac{3675}{64}\big[\langle r^2\rangle\big]^3. \tag{16}$$

The ratio of the numbers in Eqs. (12), (16) is $128/441 \sim 0.29$, showing the difference of relevant moments between to form-factor "models". Even if we took the sixth root of this number to bring it closer to unity – as experimentalists do with $\langle r^2\rangle$ to halve the relative error – the result would, at the required great precision, still epitomize the model-dependence of the results.

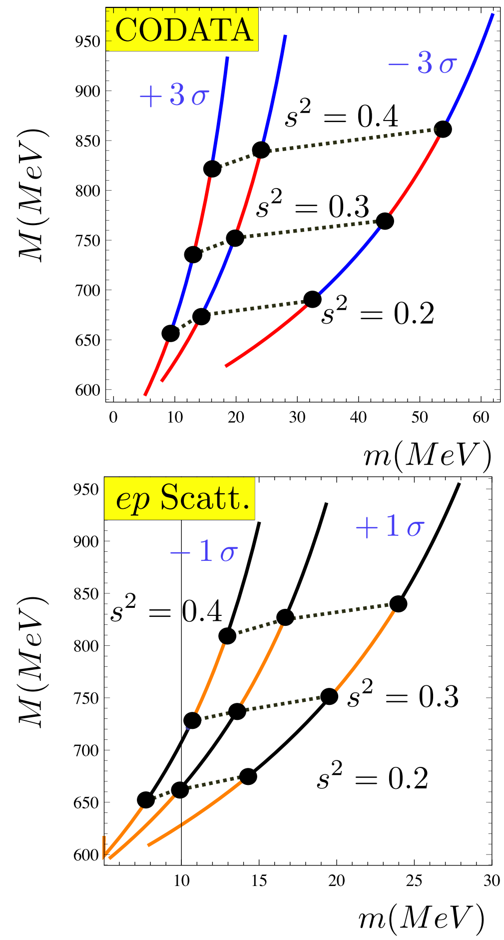

Fig. 1. Parameters M and m for which the toy model is compatible with the data, with $s^2 = \sin^2(\theta)$ varying along the curves, see Eqs. (20), (21). Top: Lyman in the μp atom and CODATA, shown for the central value and a very asymmetric $\pm 3\sigma$. Bottom: CODATA substituted for ep scattering, central value and $\pm 1\sigma$ (there is no solution for $+3\sigma$).

2.1. A toy model

The photon propagator in the time-like domain ($q^2 > 0$) has led to considerable revolutions (e.g. the discovery and interpretation of the J/Ψ), as well as interesting challenges, in particular close to its cut at $q^2 \geqslant 4m_\pi^2$. The modeling of the electric and magnetic form factors G_E and G_M of protons and neutrons in terms of dispersion relations for the photon propagator involves, literally, dozens of parameters [4]. The form-factor "toy model" I am going to discuss is not intended to compete in accuracy with the dispersive approaches, nor to be a realistic description of ep data, but only to elucidate the current discussion.

In [4], an accurate description of the theoretically-calculated 2π continuum required products of up to three poles. I parametrize $\rho(r)$ as an interpolation between the charge densities of a "ρ" single pole and a "2π" dipole:

$$\rho(r) = \frac{1}{D}\left[\frac{M^4 e^{-Mr}\cos^2(\theta)}{4\pi r} + \frac{m^5 e^{-mr}\sin^2(\theta)}{8\pi}\right],$$
$$D \equiv M^2\cos^2(\theta) + m^2\sin^2(\theta) \tag{17}$$

whose two first relevant moments are $\langle r^0\rangle = 1$ and

$$\langle r^2\rangle\big|_{\rm toy} = \frac{6}{m^2\tan^2(\theta) + M^2} + \frac{12}{m^2 + M^2\cot^2(\theta)}. \tag{18}$$

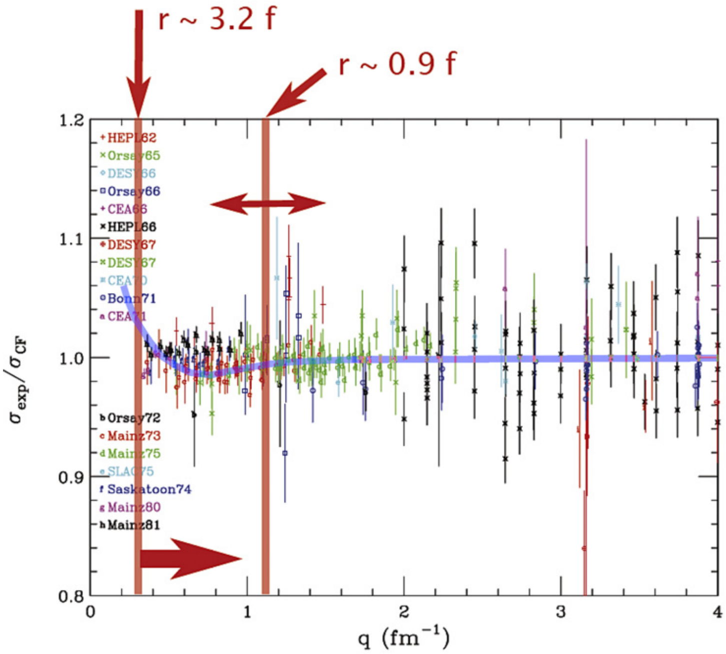

Fig. 2. Low-|**q**| data, compiled and analyzed by Sick [9]. The lines are my addition. Only $r \sim 0.9$ f is bracketed by the data, which are all to one side of $r \sim 3.2$ f. The continuous curve – drawn assuming that the absolute data normalization is not sacred – illustrates a possible shape whose corresponding $\rho_p(r)$ would have a conventional $\langle r_p^2\rangle$ and a "large" $\langle r_p^3\rangle_{(2)}$. Below $q \sim 1$ inverse fermi, notice the different averages of the Mainz81 (black) data and the rest.

To introduce the third Zemach moment, let $s \equiv \sin(\theta)$ and $c \equiv \cos(\theta)$. Then

$$\langle r^3\rangle_{(2)}\big|_{\rm toy} = \frac{3[5mM(8c^4M + 21ms^4) + 16c^2Hs^2]}{2mM(c^2M^2 + m^2s^2)^2},$$
$$H \equiv \frac{2m^5 + 4m^4M + 6m^3M^2 + 8m^2M^3 + 10mM^4 + 5M^5}{(m+M)^2}. \quad (19)$$

We can now check the compatibility of the CODATA result of Eq. (3) with the Lamb shift result of Eq. (5) in the following way. Solve the two equations

$$\langle r_p^2\rangle(\text{CODATA}) = \langle r^2\rangle\big|_{\rm toy}, \quad (20)$$
$$L_{\rm exp} = L^{\rm th}\big[\langle r^2\rangle, \langle r^3\rangle_{(2)}\big]\big|_{\rm toy} \quad (21)$$

in M and m for fixed mixing (fixed s^2). The results are shown in the top Fig. 1, while those of a similar exercise with $\langle r_p^2\rangle$(CODATA) substituted by $\langle r_p^2\rangle(ep)$ in Eq. (20) are shown in the bottom panel.

From these figures we can draw three conclusions: (1) The system of Eqs. (20), (21) is soluble only for $s^2 \geqslant 0.1$. A single-pole or single-dipole are excluded, as expected. (2) The extracted M and m are not unreasonable. M turns out to be of $\mathcal{O}(m_\rho)$, while m, which corresponds to a dipole parametrization of "everything but the ρ pole" is not a good enough simplification, a result with $m > 2m_\pi$ would have been nicer. Yukawa intuited pions in a very similar manner, but only one at a time. (3) All experimental results are compatible.

3. Conclusions

We face a choice between the following conclusions:

- The experimental results are not right.
- The relevant QED calculations are incorrect.
- There is, at extremely low energies and at the level of accuracy of the ℓp-atom experiments, "physics beyond the standard model".
- A single-dipole form factor is not adequate to the analysis of precise low-energy data.

I have argued that the last choice is the most compelling.

The theoretical and experimental results I have quoted momentarily culminate 125 years of progress in the understanding of hydrogen and its muonic sibling (I am setting $t = 0$ at the discovery date [13] of his famous "series" by the Swiss physicist Johann Jakob Balmer).

The combination of the very impressive results in Eqs. (3), (5), (6) yields a value:

$$\big[\langle r_p^3\rangle_{(2)}\big]^{1/3} = 3.32 \pm 0.22 \text{ f} \quad (22)$$

with the error dominated by the CODATA uncertainty on $\langle r_p^2\rangle$. The value in Eq. (22) looks *incredibly* large at first, but it is not so unexpected.

The result Eq. (22) is $\rho_p(r)$-independent; to be treated with due respect. Right after offering excuses, I shall break this rule. The third Zemach moment is very sensitive to the long-distance part of $\rho(r)$, compare it to the r^3 moments of Eqs. (11), (15). Suppose that $\rho(r)$ has a "core" and a "tail" contributing 50–50 to the proton's charge, and that the tail's $G_E(\mathbf{q})$ is described by a dipole. To what scale, m, to does this tail correspond? The value of $\langle r_p^3\rangle_{(2)}$ is 1/2 of the one in Eq. (15). Equate it to Eq. (22) to obtain $m \simeq 245$ MeV, tantalizingly close to the threshold of the proton form factor's cut at $2m_{\pi^\pm} \simeq 278$ MeV.

4. Discussion

Very soon after "v2" of this Letter appeared in arXiv, a preprint by Clöet and Miller was posted [14]. These authors find it impossible to extract a result as large as that of Eq. (22) from *ep* scattering data.

A crucial problem in this connection was adroitly emphasized by Sick [9]. It is very difficult to extract reliable information on

$\rho(r)$ from its Fourier transform, $G_E(\mathbf{q})^2$. The radius of convergence of the expansion in r from which one extracts $\langle r_p^2\rangle$ is so small, that one must use simulations and a continued-fraction expansion to obtain a stable, numerically-meaningful result not contaminated, for instance, by the term in $\langle r_p^4\rangle$. Clearly, if extricating $\langle r_p^2\rangle$ is delicate, the more so it is to infer $\langle r_p^3\rangle_{(2)}$ [12].

In [9] and [12] the root mean-square radius and the third Zemach moment are extracted from the same data set. It is customary to present the results for two such highly-correlated quantities as contour plots of confidence levels in a plane, to display the correlation. The authors, however, report only independent uncertainties on each individual quantity. Neither do the authors of [14] discuss the point that one is dealing with data and their correlated errors, rather than with exact cancellations in algebraic expressions.

A problem in extracting $\langle r_p^3\rangle_{(2)}$ from *ep* data is illustrated in Fig. 2, borrowed from [9]. The data amply bracket a domain around $r\sim 0.9$ f, required to measure an $\langle r_p^2\rangle^{1/2}$ of this order. Contrariwise, these data are all to one side of $r\sim 3.2$ f, the $\langle r_p^3\rangle_{(2)}^{1/3}$ scale in Eq. (22). A result based on them has to be an extrapolation of data with a large spread and a poor χ^2 per degree of freedom.

The absolute normalization of the data at small $\mathbf{q}^2$ (and their always disdained systematic errors) are a notorious hurdle [9]. Notice in particular how, below one inverse fermi, the Mainz81 data (in black) and the rest (red and green) appear to have different averages and even diverse trends. If one tolerates a few % uncertainty, the actual $G_E(\mathbf{q})$ may correspond to the shape shown in Fig. 2, reminiscent of the $\pi\pi$ contribution to $F_{1,2}$ calculated in [4]. The Fourier transform of this shape would have the customary $\langle r_p^2\rangle$ but a "surplisingly" large $\langle r_p^3\rangle_{(2)}$.

Theoretical estimates of $\langle r_p^3\rangle_{(2)}$ would be interesting. In their study [4], Belushkin et al. find that the 2π continuum is a very significant contribution, rising dramatically below its scale, $|\mathbf{q}| = 2m_\pi \sim 0.71$ f^{-1}, and requiring for its parametrization products of up to three poles. I have argued that the clash between muonic-hydrogen and *ep* data may only be apparent. The issue may be decisively settled only via new *ep* scattering experiments, or via a reanalysis of current data, perhaps with a "large" third Zemach moment as a constraint.

In a third version of their paper, the authors of [14] cogently argue that a third Zemach moment as large as that of Eq. (22) is very hard to reconcile with the *ep* data. This point deserves further discussion, which I postpone. Notice that the data in Fig. 2, relative to a straight horizontal line at $\sigma_{\rm exp}/\sigma_{\rm CF}=1$, have, with no further ado, a $\chi^2/\text{dof}\sim 1.65$ for 310 degrees of freedom [9], that is a p-value (the probability of a result equally or less compatible with the hypothesis) of 3.9×10^{-12}. This casts doubt even on the corresponding extracted value of the mean square radius, which could easily be equally compatible with a number half way between the CODATA and muonichydrogen results.

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